THE GRAHAM CONJECTURE IMPLIES THE ERDÖS-TURÁN CONJECTURE

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ABSTRACT. Erdös and Turán once conjectured that any set $A \subset \mathbb{N}$ with $\sum_{a\in A} 1/a = \infty$ should contain infinitely many progressions of arbitrary length $k \geq 3$. For the two-dimensional case Graham conjectured that if $B \subset \mathbb{N}\times\mathbb{N}$ satisfies

$$\sum_{(x,y)\in B} \frac{1}{x^2+y^2} = \infty,$$

then for any $s \geq 2$, B contains an $s\times s$ axes-parallel grid. In this paper it is shown that if the Graham conjecture is true for some $s \geq 2$, then the Erdös-Turán conjecture is true for $k = 2s-1$.

1. INTRODUCTION

One famous conjecture of Erdös and Turán [2] asserts that any set $A \subset \mathbb{N}$ with $\sum_{a\in A} 1/a = \infty$ should contain infinitely many progressions of arbitrary length $k \geq 3$. There are two important progresses towards this direction due to Szemerédi [7] and Green and Tao [5] respectively, which assert that if A has positive upper density or A is the set of the prime numbers, then A contains infinitely many progressions of arbitrary length.

If one considers the similar question in the two-dimensional plane, Graham [4] conjectured that if $B \subset \mathbb{N}\times\mathbb{N}$ satisfies

$$\sum_{(x,y)\in B} \frac{1}{x^2+y^2} = \infty,$$

then B contains the four vertices of an axes-parallel square. More generally, for any $s \geq 2$ it should be true that B contains an $s\times s$ axes-parallel grid. Furstenberg and Katznelson [3] proved the two-dimensional Szemerédi theorem, that is, any set $B \subset \mathbb{N}\times\mathbb{N}$ with positive upper density contains an $s\times s$ axes-parallel grid. In another words, such a set B contains any finite pattern.

The purpose of this paper is to show that if the Graham conjecture is true, then the Erdös-Turán conjecture is also true.

2. THE GRAHAM CONJECTURE IMPLIES THE ERDÖS-TURÁN CONJECTURE

Suppose that the Erdös-Turán conjecture is false for $k = 3$. Then there exists a set

$$A = \{a_1 < a_2 < a_3 < \cdots\} \subset \mathbb{N}$$

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with $\sum_{n\in\mathbb{N}} 1/a_n = \infty$ such that A contains no arithmetic progression of length 3. Define a set $B \subset \mathbb{N}\times\mathbb{N}$ by

$$B = \Big\{(a_n + m, m) : n \in \mathbb{N}, m \in \mathbb{N}\Big\}.$$

Then

$$\begin{aligned}
\sum_{(x,y)\in B} \frac{1}{x^2+y^2} &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \frac{1}{(a_n+m)^2+m^2} \\
&\geq \sum_{n\in\mathbb{N}}\sum_{m=1}^{a_n} \frac{1}{(a_n+m)^2+m^2} \\
&\geq \sum_{n\in\mathbb{N}} \frac{a_n}{(a_n+a_n)^2+a_n^2} \\
&= \sum_{n\in\mathbb{N}} \frac{1}{5a_n} \\
&= \infty.
\end{aligned}$$

In the sequel we indicate that B contains no square and argue it by contradiction. This would mean that the Graham conjecture is false for $s = 2$. Suppose that for some $n, m, l \in \mathbb{N}$, B contains a square of the following form:

$$\begin{array}{ll}
(a_n + m, m + l), & (a_n + m + l, m + l), \\
(a_n + m, m), & (a_n + m + l, m).
\end{array}$$

It follows easily from the construction of B that $a_n - l, a_n, a_n + l \in A$, which yields a contradiction since A contains no arithmetic progression of length 3 according to the initial assumption.

Similarly, if the Graham conjecture is true for some $s \geq 2$, then the Erdös-Turán conjecture is true for $k = 2s - 1$. The interested reader can easily provide a proof.

3. Concluding Remarks

Let $r(k, N)$ be the maximal cardinality of a subset A of $\{1, 2, \ldots, N\}$ which is free of k-term arithmetic progressions. Behrend [1] and Rankin [6] had shown that

$$r(k, N) \geq N \cdot \exp(-c(\log N)^{1/(k-1)}).$$

Similarly, let $\widetilde{r}(s, N)$ be the maximal cardinality of a subset B of $\{1, 2, \ldots, N\}^2$ which is free of $s \times s$ axes-parallel grids. For any set $A \subset \{1, 2, \ldots, N\}$, define

$$\Theta(A) = \{(a + m, m) : a \in A, m = 1, 2, \ldots, N\} \subset \{1, 2, \ldots, 2N\}^2.$$

Following the discussion in Section 2, one can easily deduce that if A is free of $2s-1$ term of arithmetic progression, then $\Theta(A)$ is free of $s \times s$ axes-parallel grid. Hence

$$\begin{aligned}
\widetilde{r}(s, 2N) &\geq r(2s - 1, N) \cdot N \\
&\geq N^2 \exp(-c(\log N)^{1/(2s-2)}).
\end{aligned}$$

We end this paper with a question. Does the Erdös-Turán conjecture imply the Graham conjecture?

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